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Borel reducibility and finitely Hölder(α) embeddability

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Abstract

Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of pseudo-metric spaces, $p \geq 1$. For $x, y \in \prod_{n\in\mathbb{N}} X_n$, let $(x, y) \in E((X_n)_{n\in\mathbb{N}}; p) \Leftrightarrow \sum_{n\in\mathbb{N}} d_n(x(n), y(n))^p < +\infty$. For Borel reducibility between equivalence relations $E((X_n)_{n\in\mathbb{N}}; p)$, we show it is closely related to finitely Hölder(α) embeddability between pseudo-metric spaces.

Keywords: Borel reducibility, Hölder(α) embeddability, finitely Hölder(α) embeddability

1. Introduction

A topological space is called a *Polish space* if it is homeomorphic to a separable complete metric space. Let X, Y be Polish spaces and E, F equivalence relations on X, Y respectively. A *Borel reduction* from E to F is a Borel function $\theta : X \to Y$ such that

$$(x, y) \in E \iff (\theta(x), \theta(y)) \in F$$

for all $x, y \in X$. We say that E is *Borel reducible* to F, denoted $E \leq_B F$, if there is a Borel reduction from E to F. If $E \leq_B F$ and $F \leq_B E$, we say that E and F are *Borel bireducible* and denote $E \sim_B F$. We refer to [1] and [5] for background on Borel reducibility.

It was proved by R. Dougherty and G. Hjorth [4] that, for $p, q \geq 1$,

$$\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_q \iff p \leq q.$$

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The equivalence relation $\mathbb{R}/\ell_p$ was extended to so called ℓ_p-like equivalence relations in [3]. Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of pseudo-metric spaces, $p \geq 1$. For $x, y \in \prod_{n\in\mathbb{N}} X_n$, $(x, y) \in E((X_n)_{n\in\mathbb{N}}; p) \Leftrightarrow \sum_{n\in\mathbb{N}} d_n(x(n), y(n))^p < +\infty$.

A special case concerning separable Banach spaces was investigated in [2]. It was showed in [2] that Borel reducibility between this kind of equivalence relations is related to the existence of Hölder(α) embeddings. In this paper, we introduce the notion of C-finitely Hölder(α) embeddability, and generalize the connection between Borel reducibility and finitely Hölder(α) embeddability to a rather general type of metric spaces.

2. ℓ_p-like equivalence relations on pseudo-metric spaces

Definition 2.1. *Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of pseudo-metric spaces, $p \geq 1$. We define an equivalence relation $E((X_n, d_n)_{n\in\mathbb{N}}; p)$ on $\prod_{n\in\mathbb{N}} X_n$ by*

$$(x, y) \in E((X_n, d_n)_{n\in\mathbb{N}}; p) \iff \sum_{n\in\mathbb{N}} d_n(x(n), y(n))^p < +\infty$$

for $x, y \in \prod_{n\in\mathbb{N}} X_n$. We call it an ℓ_p-like equivalence relation.

If $(X_n, d_n) = (X, d)$ for every $n \in \mathbb{N}$, we write $E((X, d); p) = E((X_n, d_n)_{n\in\mathbb{N}}; p)$ for the sake of brevity. If there is no danger of confusion, we simply write $E((X_n)_{n\in\mathbb{N}}; p)$ and $E(X; p)$ instead of $E((X_n, d_n)_{n\in\mathbb{N}}; p)$ and $E((X, d); p)$.

Definition 2.2. *If X is a Polish space, d is a Borel pseudo-metric on X, we say (X, d) is a Borel pseudo-metric space.*

Let (Y_n, δ_n), $n \in \mathbb{N}$ be a sequence of pseudo-metric spaces, $y^* \in \prod_{n\in\mathbb{N}} Y_n$. For $q \geq 1$, we denote by $\ell_q((Y_n)_{n\in\mathbb{N}}, y^*)$ the pseudo-metric space whose underlying space is

$$\left\{ y \in \prod_{n\in\mathbb{N}} Y_n : \sum_{n\in\mathbb{N}} \delta_n(y(n), y^*(n))^q < +\infty \right\},$$

with the pseudo-metric

$$\delta_q(x, y) = \left(\sum_{n\in\mathbb{N}} \delta_n(x(n), y(n))^q \right)^{\frac{1}{q}}.$$

Theorem 2.3. *Let (Y,δ) be a Borel pseudo-metric space, $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$ a sequence of Borel subsets of Y, and let $(X_n,d_n), n \in \mathbb{N}$ be a sequence of Borel pseudo-metric spaces, $p,q \in [1,+\infty)$. If there are $A,C,D>0$, a sequence of Borel maps $T_n : X_n \to \ell_q((Y_n)_{n\in\mathbb{N}}, y^*)$ for some $y^* \in \prod_{n\in\mathbb{N}} Y_n$ and two sequences of non-negative real numbers $\varepsilon_n, \eta_n, n \in \mathbb{N}$ such that*

(1) $\sum_{n\in\mathbb{N}} \varepsilon_n^p < +\infty, \sum_{n\in\mathbb{N}} \eta_n^q < +\infty$;
(2) $d_n(u,v) < \varepsilon_n \Rightarrow \delta_q(T_n(u),T_n(v)) < \eta_n$;
(3) $d_n(u,v) \geq C \Rightarrow \delta_q(T_n(u),T_n(v)) \geq D$;
(4) $\varepsilon_n \leq d_n(u,v) < C \Rightarrow A^{-1} d_n(u,v)^{\frac{p}{q}} \leq \delta_q(T_n(u),T_n(v)) \leq A d_n(u,v)^{\frac{p}{q}}$.

Then we have
$$E((X_n)_{n\in\mathbb{N}};p) \leq_B E((Y_n)_{n\in\mathbb{N}};q).$$

Proof. Fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$ such that $m \leq \langle n,m\rangle$ for each $n,m \in \mathbb{N}$. Note that $T_n(u)(m) \in Y_m \subseteq Y_{\langle n,m\rangle}$ for every $u \in X_n$. We define $\theta : \prod_{n\in\mathbb{N}} X_n \to \prod_{k\in\mathbb{N}} Y_k$ by
$$\theta(x)(\langle n,m\rangle) = T_n(x(n))(m)$$
for $x \in \prod_{n\in\mathbb{N}} X_n$ and $n,m \in \mathbb{N}$. It is easy to see that θ is Borel. By the definition we have
$$\begin{array}{rl} & \sum_{n,m\in\mathbb{N}} \delta(\theta(x)(\langle n,m\rangle), \theta(y)(\langle n,m\rangle))^q \\ = & \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \delta(T_n(x(n))(m), T_n(y(n))(m))^q \\ = & \sum_{n\in\mathbb{N}} \delta_q(T_n(x(n)), T_n(y(n)))^q\,. \end{array}$$

For $x,y \in \prod_{n\in\mathbb{N}} X_n$, we split $\mathbb{N}$ into three sets
$$I_1 = \{n \in \mathbb{N} : d_n(x(n),y(n)) < \varepsilon_n\},$$
$$I_2 = \{n \in \mathbb{N} : d_n(x(n),y(n)) \geq C\},$$
$$I_3 = \{n \in \mathbb{N} : \varepsilon_n \leq d_n(x(n),y(n)) < C\}.$$
From (2) we have
$$\sum_{n\in I_1} d_n(x(n),y(n))^p < \sum_{n\in I_1} \varepsilon_n^p \leq \sum_{n\in\mathbb{N}} \varepsilon_n^p < +\infty,$$
$$\sum_{n\in I_1} \delta_q(T_n(x(n)),T_n(y(n)))^q < \sum_{n\in I_1} \eta_n^q \leq \sum_{n\in\mathbb{N}} \eta_n^q < +\infty;$$

denote $|I_2|$ the cardinal of I_2, from (3) we have

$$\sum_{n\in I_2} d_n(x(n),y(n))^p \geq C^p|I_2|,$$

$$\sum_{n\in I_2} \delta_q(T_n(x(n)),T_n(y(n)))^q \geq D^q|I_2|;$$

and from (4) we have

$$A^{-q}\sum_{n\in I_3} d_n(x(n),y(n))^p \leq \sum_{n\in I_3} \delta_q(T_n(x(n)),T_n(y(n)))^q \leq A^q\sum_{n\in I_3} d_n(x(n),y(n))^p.$$

Therefore,

$$\begin{array}{cl} & (x,y)\in E((X_n)_{n\in\mathbb{N}};p) \\ \iff & \sum_{n\in\mathbb{N}} d_n(x(n),y(n))^p < +\infty \\ \iff & |I_2|<\infty,\ \sum_{n\in I_3} d_n(x(n),y(n))^p < +\infty \\ \iff & |I_2|<\infty,\ \sum_{n\in I_3} \delta_q(T_n(x(n)),T_n(y(n)))^q < +\infty \\ \iff & \sum_{n\in\mathbb{N}} \delta_q(T_n(x(n)),T_n(y(n)))^q < +\infty \\ \iff & \sum_{n,m\in\mathbb{N}} \delta(\theta(x)(\langle n,m\rangle),\theta(y)(\langle n,m\rangle))^q < +\infty \\ \iff & (\theta(x),\theta(y))\in E((Y_k)_{k\in\mathbb{N}};q). \end{array}$$

It follows that $E((X_n)_{n\in\mathbb{N}};p) \leq_B E((Y_n)_{n\in\mathbb{N}};q)$. ☐

Corollary 2.4. *If all (X_n,d_n)'s are separable, then the sequence $\eta_n, n\in\mathbb{N}$ and clause (2) in Theorem 2.3 can be omitted.*

Proof. By Zorn's lemma, we can find a set $S_n\subseteq X_n$ for each n such that

(i) $\forall r,s\in S_n(r\neq s\to d_n(r,s)\geq\varepsilon_n)$;
(ii) $\forall u\in X_n\exists s\in S_n(d_n(u,s)<\varepsilon_n)$.

Since X_n is separable, S_n is countable. So we can enumerate S_n by $(s^n_m)_{m\in\mathbb{N}}$. Define $T'_n: X_n\to\ell_q((Y_n)_{n\in\mathbb{N}},y^*)$ by $T'_n(u)=T_n(s^n_{m(u)})$ where $m(u)$ is the least m such that $d_n(u,s^n_m)<\varepsilon_n$. It is easy to see that each T'_n is Borel.

Without loss of generality, we may assume that $5\varepsilon_n<C$. Now denote $\varepsilon'_n=3\varepsilon_n, \eta'_n=A(5\varepsilon_n)^{\frac{p}{q}}$ and $A'=3^{\frac{p}{q}}A, C'=C-2\varepsilon_n, D'=\min\left\{D, A^{-1}\left(\frac{C}{5}\right)^{\frac{p}{q}}\right\}$. We check that $\varepsilon'_n,\eta'_n,A',C'$ and D' meet clauses (1)–(4) in Theorem 2.3 as follows:

(1) $\sum_{n\in\mathbb{N}}(\varepsilon'_n)^p=3^p\sum_{n\in\mathbb{N}}\varepsilon_n^p<+\infty$, $\sum_{n\in\mathbb{N}}(\eta'_n)^q=5^pA^q\sum_{n\in\mathbb{N}}\varepsilon_n^p<+\infty$.

(2) If $d_n(u,v) < \varepsilon'_n$, then $d_n(s^n_{m(u)}, s^n_{m(v)}) < 5\varepsilon_n < C$. Note that $s^n_{m(u)} = s^n_{m(v)}$ or $d_n(s^n_{m(u)}, s^n_{m(v)}) \geq \varepsilon_n$. So by clause (4) in Theorem 2.3, we have

$$\delta_q(T'_n(u), T'_n(v)) = \delta_q(T_n(s^n_{m(u)}), T_n(s^n_{m(v)})) \leq A d_n(s^n_{m(u)}, s^n_{m(v)})^{\frac{p}{q}} < \eta'_n.$$

(3) If $d_n(u,v) \geq C'$, then $d_n(s^n_{m(u)}, s^n_{m(v)}) \geq C - 4\varepsilon_n \geq \varepsilon_n$. For $\varepsilon_n \leq d_n(s^n_{m(u)}, s^n_{m(v)}) < C$, we have

$$\begin{aligned} \delta_q(T'_n(u), T'_n(v)) &= \delta_q(T_n(s^n_{m(u)}), T_n(s^n_{m(v)})) \\ &\geq A^{-1} d_n(s^n_{m(u)}, s^n_{m(v)})^{\frac{p}{q}} \geq A^{-1}(C - 4\varepsilon_n)^{\frac{p}{q}} \\ &\geq A^{-1}\left(\frac{C}{5}\right)^{\frac{p}{q}} \geq D'. \end{aligned}$$

And for $d_n(s^n_{m(u)}, s^n_{m(v)}) \geq C$, we have

$$\delta_q(T'_n(u), T'_n(v)) = \delta_q(T_n(s^n_{m(u)}), T_n(s^n_{m(v)})) \geq D \geq D'.$$

(4) If $\varepsilon'_n \leq d_n(u,v) < C'$, then $\varepsilon_n \leq d_n(s^n_{m(u)}, s^n_{m(v)}) < C$ and

$$\frac{1}{3} d_n(u,v) \leq d_n(u,v) - 2\varepsilon_n < d_n(s^n_{m(u)}, s^n_{m(v)}) < d_n(u,v) + 2\varepsilon_n \leq 3 d_n(u,v).$$

Since

$$A^{-1} d_n(s^n_{m(u)}, s^n_{m(v)})^{\frac{p}{q}} \leq \delta_q(T_n(s^n_{m(u)}), T_n(s^n_{m(v)})) \leq A d_n(s^n_{m(u)}, s^n_{m(v)})^{\frac{p}{q}},$$

it follows that

$$(A')^{-1} d_n(u,v)^{\frac{p}{q}} \leq \delta_q(T'_n(u), T'_n(v)) \leq A' d_n(u,v)^{\frac{p}{q}}.$$

$\square$

3. On separable pseudo-metric spaces

For the rest of this paper, we focus on such $E((X_n, d_n)_{n\in\mathbb{N}}; p)$ that all (X_n, d_n)'s are separable Borel pseudo-metric spaces.

Let $S_n = \{s^n_m : m \in \mathbb{N}\}$ be a countable dense subset of X_n. We may assume that $d_n(s^n_m, s^n_k) > 0$ for $m \neq k$, i.e. (S_n, d_n) is a countable metric space. For $u \in X_n$, let $m_n(u) = \min\{m : d_n(u, s^n_m) < 2^{-n}\}$ and $\vartheta : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} D_n$ as $\vartheta(x)(n) = s^n_{m_n(u)}$. Since $\sum_{n\in\mathbb{N}} d_n(x(n), \vartheta(x)(n))^p <$

$\sum_{n\in\mathbb{N}} 2^{-np} < +\infty$, we have $(x, \vartheta(x)) \in E((X_n)_{n\in\mathbb{N}}; p)$. Thus ϑ is a Borel reduction of $E((X_n)_{n\in\mathbb{N}}; p)$ to $E((S_n)_{n\in\mathbb{N}}; p)$. So $E((X_n)_{n\in\mathbb{N}}; p) \sim_B E((S_n)_{n\in\mathbb{N}}; p)$. Now let $(\overline{S_n}, \overline{d_n})$ be the completion of (S_n, d_n). Since $(\overline{S_n}, \overline{d_n})$ is a Polish space, by the same arguments, we have

$$E((\overline{S_n})_{n\in\mathbb{N}}; p) \sim_B E((S_n)_{n\in\mathbb{N}}; p) \sim_B E((X_n)_{n\in\mathbb{N}}; p).$$

Therefore, from now on, we may assume that all (X_n, d_n)'s are separable complete metric space.

Definition 3.1. *Let (X, d) be a separable complete metric space, $(F_n)_{n\in\mathbb{N}}$ a sequence of finite subsets of X. If $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$ and $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X, then we denote*

$$F(X; p) = E((F_n)_{n\in\mathbb{N}}; p).$$

The following lemma shows that, under Borel bireducibility, $F(X; p)$ is independent to the choice of $(F_n)_{n\in\mathbb{N}}$.

Lemma 3.2. *Let (X, d) be a separable complete metric space, and let $(F_n)_{n\in\mathbb{N}}$ and $(F'_n)_{n\in\mathbb{N}}$ be two sequences of finite subsets of X satisfying that*

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots, \quad F'_0 \subseteq F'_1 \subseteq \cdots \subseteq F'_n \subseteq \cdots,$$

and both $\bigcup_{n\in\mathbb{N}} F_n$ and $\bigcup_{n\in\mathbb{N}} F'_n$ are dense in X. Then for each $p \geq 1$we have

$$E((F_n)_{n\in\mathbb{N}}; p) \sim_B E((F'_n)_{n\in\mathbb{N}}; p).$$

Proof. It will suffice to show that $E((F_n)_{n\in\mathbb{N}}; p) \leq_B E((F'_n)_{n\in\mathbb{N}}; p)$. For $k \in \mathbb{N}$, let $\gamma_k = \min\{d(u, v) : u, v \in F_k, u \neq v\}$. Note that $\bigcup_{n\in\mathbb{N}} F'_n$ is dense in X. For $u \in F_k$, we can find a $T_k(u) \in \bigcup_{n\in\mathbb{N}} F'_n$ such that $d(u, T_k(u)) < \gamma_k/4$. Then for distinct $u, v \in F_k$ we have

$$\frac{1}{2}d(u, v) \leq d(u, v) - \gamma_k/2 < d(T_k(u), T_k(v)) < d(u, v) + \gamma_k/2 \leq 2d(u, v).$$

Since F_k is finite, there is n_k such that $T_k(u) \in F'_{n_k}$ for each $u \in F_k$. We may assume that $(n_k)_{k\in\mathbb{N}}$ is strictly increasing. Fix a point $u_0 \in F'_0 \subseteq F'_n$. We define $\theta : \prod_{n\in\mathbb{N}} F_n \to \prod_{n\in\mathbb{N}} F'_n$ by

$$\theta(x)(n) = \begin{cases} T_k(x(k)), & n = n_k \\ u_0, & \text{otherwise.} \end{cases}$$

Then for $x, y \in \prod_{n\in\mathbb{N}} F_n$ we have

$$\frac{1}{2^p}\sum_{k\in\mathbb{N}} d(x(k), y(k))^p \le \sum_{n\in\mathbb{N}} d(\theta(x)(n), \theta(y)(n))^p \le 2^p \sum_{k\in\mathbb{N}} d(x(k), y(k))^p,$$

It follows that θ is a Borel reduction of $E((F_n)_{n\in\mathbb{N}}; p)$ to $E((F'_n)_{n\in\mathbb{N}}; p)$. $\square$

Remark 3.3. *We can see that $E(X; p) \sim_B F(X; p)$ when X is compact. But whether it is always true for every separable complete metric space? We do not know the answer.*

Definition 3.4. *For two metric spaces $(X, d), (X', d')$ and $\alpha > 0$. We say that X Hölder(α) embeds into X' if there exist $A > 0$ and $T : X \to X'$ such that, for $u, v \in F$,*

$$A^{-1}d(u, v)^\alpha \le d'(T(u), T(v)) \le Ad(u, v)^\alpha.$$

Theorem 2.3 gives the following result.

Remark 3.5. *Let X, Y be two separable complete metric spaces, $p, q \in [1, +\infty)$. If X Hölder($\frac{p}{q}$) embeds into $\ell_q(Y, y^*)$ for some $y^* \in Y^{\mathbb{N}}$, then we have $E(X; p) \le_B E(Y; q)$.*

In next section, we present a necessary condition of $E(X; p) \le_B E(Y; q)$ which will be named finitely Hölder($\frac{p}{q}$) embeddability.

4. Finitely Hölder(α) embeddability

A weak version of the following lemma is due to R. Dougherty and G. Hjorth [4]. For self-contain reason, we present a proof for it.

Lemma 4.1. *Let $(Y_n, \delta_n), n \in \mathbb{N}$ be a a sequence of separable complete metric space, $p, q \in [1, +\infty)$, and let $(Z_n, d_n), n \in \mathbb{N}$ be a sequence of finite metric spaces. Assume that $E((Z_n)_{n\in\mathbb{N}}; p) \le_B E((Y_n)_{n\in\mathbb{N}}; q)$. Then there exist strictly increasing sequences of natural numbers $(b_j)_{j\in\mathbb{N}}, (l_j)_{j\in\mathbb{N}}$ and $T_j : Z_{b_j} \to \prod_{n=l_j}^{l_{j+1}-1} Y_n$ such that, for $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$, we have*

$$(x, y) \in E((Z_{b_j}, d_{b_j})_{j\in\mathbb{N}}; p) \iff \sum_{j\in\mathbb{N}} \delta_q(T_j(x(j)), T_j(y(j)))^q < +\infty,$$

where $\delta_q(r, s) = (\sum_{n=l_j}^{l_{j+1}-1} \delta_n(r(n), s(n))^q)^{\frac{1}{q}}$ for $r, s \in \prod_{n=l_j}^{l_{j+1}-1} Y_n$.

Proof. The proof is modified from the proof of [4] Theorem 2.2, Claim (i)–(iii).

Denote $Z = \prod_{n\in\mathbb{N}} Z_n$. Assume that θ is a Borel reduction of $E((Z_n)_{n\in\mathbb{N}}; p)$ to $E((Y_n)_{n\in\mathbb{N}}; q)$. For each finite sequence t we denote $l(t)$ the length of t; if $t \in \prod_{i<l(t)} Z_i$, let $N_t = \{z \in Z : z(i) = t(i)\,(i < l(t))\}$.

Claim (i). For $j, k \in \mathbb{N}$, there exist $l \in \mathbb{N}$ and $s^* \in \prod_{i=k}^{k+l(s^*)-1} Z_i$ and a comeager set $D \subseteq Z$ such that, for all $x, \hat{x} \in D$, if we have $x = rs^*y$ and $\hat{x} = \hat{r}s^*y$ for some $r, \hat{r} \in \prod_{i<k} Z_i$ and $y \in \prod_{i\geq k+l(s^*)} Z_i$, then

$$\sum_{n\geq l} \delta_n(\theta(x)(n), \theta(\hat{x})(n))^q < 2^{-j}.$$

Proof. For $l \in \mathbb{N}$, we define a function $F_l : Z \to \mathbb{R}$ by

$$F_l(x) = \max\left\{\sum_{n\geq l} \delta_n(\theta(z)(n), \theta(\hat{z})(n))^q : z(i) = \hat{z}(i) = x(i)\,(i \geq k)\right\}.$$

For each x, there are only finitely many pairs $z, \hat{z}$ satisfying $z(i) = \hat{z}(i) = x(i)\,(i \geq k)$. For each such pair we have $(z, \hat{z}) \in E((Z_n)_{n\in\mathbb{N}}; p)$, so $(\theta(z), \theta(\hat{z})) \in E((Y_n)_{n\in\mathbb{N}}; q)$. Thus $\lim_{l\to\infty} \sum_{n\geq l} \delta_n(\theta(z)(n), \theta(\hat{z})(n))^q = 0$. Hence $F_l(x) < +\infty$ for all l and $\lim_{l\to\infty} F_l(x) = 0$. Therefore, by the Baire category theorem, there exists an l such that $\{x : F_l(x) < 2^{-j}\}$ is not meager. By F is Borel, this set has the property of Baire, so there is an open set $O \neq \emptyset$ on which it is relatively comeager.

Find an $N_t \subseteq O$ for some finite sequence t with $l(t) \geq k$. Let $t = r^*s^*$ where $l(r^*) = k$. Since $F_l(x)$ does not depend on the first k coordinates of x, we have $\{x : F_l(x) < 2^{-j}\}$ is also relatively comeager in N_{rs^*} for all $r \in \prod_{i<k} Z_i$. Let D be a comeager set such that $F_l(x) < 2^{-j}$ whenever $x \in D \cap N_{rs^*}$ for any r of length k. Now the conclusion of the claim follows from the definition of F_l. Claim (i) $\square$

By [6] Theorem (5.38), there is a dense G_δ set $C \subseteq Z$ such that $\theta \upharpoonright C$ is continuous.

Claim (ii). For $j, k, l \in \mathbb{N}$, there exists a finite sequence $s^{**} \in \prod_{i=k}^{k+l(s^{**})-1} Z_i$ such that, for all $x, \hat{x} \in C$, if we have $x = rs^{**}y$ and $\hat{x} = rs^{**}\hat{y}$ for some $r \in \prod_{i<k} Z_i$ and $y, \hat{y} \in \prod_{i\geq k+l(s^{**})} Z_i$, then

$$\sum_{n<l} \delta_n(\theta(x)(n), \theta(\hat{x})(n))^q < 2^{-j}.$$

Furthermore, if G is a given dense open subset of Z, then s^{**} can be chosen such that $N_{rs^{**}} \subseteq G$ for all $r \in \prod_{i<k} Z_i$.

Proof. Since $\prod_{i<k} Z_i$ is a finite set, we may enumerate its elements as $r_0, r_1, \cdots, r_{M-1}$. We construct finite sequences $t_0, t_1, \cdots, t_M$ as follows.

Let $t_0 = \emptyset$. Suppose that $m < M$ and we have constructed a finite sequence $t_m \in \prod_{i=k}^{k+l(t_m)-1} Z_i$. The basic open set $N_{r_m t_m}$ must meet the comeager set C, so we can pick a $w \in C \cap N_{r_m t_m}$. Since θ is continuous on C and δ_n is continuous on Y_n^2, we can find a neighborhood O of w such that, for all $x, \hat{x} \in C \cap O$, $\sum_{n<l} \delta_n(\theta(x)(n), \theta(\hat{x})(n))^q < 2^{-j}$. Find an $N_{r_m t'_m} \subseteq N_{r_m t_m} \cap O$, then $t_m \subseteq t'_m$. Since G is open dense, we can further extend t'_m to get t_{m+1} such that $N_{r_m t_{m+1}} \subseteq G$. Once the sequences $t_m\,(m \leq M)$ are constructed, $s^{**} = t_M$ fulfills the requirements. Claim (ii) $\square$

We now repeatedly apply Claims (i) and (ii) to define natural numbers $b_0 < b_1 < b_2 < \cdots$ and $l_0 < l_1 < l_2 < \cdots$, finite sequences $(s_j)_{j\in\mathbb{N}}$ and dense open sets $D_i^j \subseteq Z\,(i, j \in \mathbb{N})$ as follows.

Let $b_0 = l_0 = 0$. Suppose we have constructed $b_j, l_j, D_i^{j'}\,(j' < j)$. Applying Claim (i) for this j with $k = b_j + 1$, we get l_{j+1}, a finite sequence s_j^* and a comeager set D^j satisfying the conclusion of Claim (i). Let $D_0^j \supseteq D_1^j \supseteq D_2^j \supseteq \cdots$ be dense open sets of Z such that $\bigcap_{i\in\mathbb{N}} D_i^j \subseteq D^j \cap C$. Now apply Claim (ii) for j with $k = b_j + 1 + l(s_j^*), l = l_{j+1}$ and $G = \bigcap_{j'<j} D_j^{j'}$ to get s_j^{**}. We set $s_j = s_j^* s_j^{**}$ and $b_{j+1} = b_j + l(s_j) + 1$.

Denote $Z' = \prod_{j\in\mathbb{N}} Z_{b_j}$ and define $h : Z' \to Z$ by

$$h(x) = \langle x(0)\rangle s_0 \langle x(1)\rangle s_1 \langle x(2)\rangle s_2 \cdots.$$

Since $s_j = s_j^* s_j^{**}$, $h(x)$ has the form $r s_j^* y$ where $l(r) = b_j + 1$, and also has the form $r s_j^{**} y$ where $l(r) = b_j + l(s^*) + 1$. Therefore, Claim (ii) for s_j^{**} gives $h(x) \in G = \bigcap_{j'<j} D_j^{j'}$. Hence, for any j, we have $h(x) \in D_i^j$ for $i > j$, so $h(x) \in D^j \cap C$. Therefore, Claims (i) and (ii) imply that, for any $x, \hat{x} \in Z'$:

(1) if $x(b_i) = \hat{x}(b_i)\,(i > j)$, then $\sum_{n \geq l_{j+1}} \delta_n(\theta(h(x))(n), \theta(h(\hat{x}))(n))^q < 2^{-j}$;
(2) if $x(b_i) = \hat{x}(b_i)\,(i \leq j)$, then $\sum_{n < l_{j+1}} \delta_n(\theta(h(x))(n), \theta(h(\hat{x}))(n))^q < 2^{-j}$.

Fix a point $u_0 \in Z_0 \subseteq Z_{b_i}$. For $j \in \mathbb{N}$ we define $T_j : Z_{b_j} \to \prod_{n=l_j}^{l_{j+1}-1} Y_n$ by

$$T_j(w) = \theta(h(\langle u_0, \cdots, u_0, w, u_0, u_0, \cdots\rangle)) \upharpoonright [l_j, l_{j+1})$$

with j u_0's before v. Let $\theta' : Z \to \prod_{n\in\mathbb{N}} Y_n$,

$$\theta'(x) = T_0(x(0)) T_1(x(1)) T_2(x(2)) \cdots.$$

Next claim shows that θ' is a Borel reduction of $E((Z_{b_j}, d_{b_j})_{j\in\mathbb{N}}; p)$ to $E((Y_n)_{n\in\mathbb{N}}; q)$.

Claim (iii). For all $x, \hat{x} \in \prod_{j\in\mathbb{N}} Z_{b_j}$, we have

$$(x, \hat{x}) \in E((Z_{b_j}, d_{b_j})_{j\in\mathbb{N}}; p) \iff (\theta'(x), \theta'(\hat{x})) \in E((Y_n)_{n\in\mathbb{N}}; q).$$

Proof. Note that

$$\begin{aligned}(x, \hat{x}) \in E((Z_{b_j}, d_{b_j})_{j\in\mathbb{N}}; p) &\iff (h(x), h(\hat{x})) \in E((Z_n, d_n)_{n\in\mathbb{N}}; p) \\ &\iff (\theta(h(x)), \theta(h(\hat{x}))) \in E((Y_n)_{n\in\mathbb{N}}; q).\end{aligned}$$

It will suffice to show that $(\theta(h(x)), \theta'(x)) \in E((Y_n)_{n\in\mathbb{N}}; q)$ for any $x \in Z'$.

For any $x \in Z'$ and $j \in \mathbb{N}$, define $e_j(x), e'_j(x) \in Z'$ by

$$e_j(x)(i) = \begin{cases} x(i), & i = j \\ u_0, & i \neq j; \end{cases} \qquad e'_j(x)(i) = \begin{cases} x(i), & i \leq j \\ u_0, & i > j. \end{cases}$$

By (1) for $j-1$ and (2), we have

$$\sum_{n\geq l_j} \delta_n(\theta(h(e_j(x)))(n), \theta(h(e'_j(x)))(n))^q < 2^{-(j-1)},$$

$$\sum_{n<l_{j+1}} \delta_n(\theta(h(x))(n), \theta(h(e'_j(x)))(n))^q < 2^{-j}.$$

Thus we have

$$\begin{aligned} & \textstyle\sum_{n=l_j}^{l_{j+1}-1} \delta_n(\theta(h(x))(n), \theta(h(e_j(x)))(n))^q \\ \leq\ & \textstyle\sum_{n=l_j}^{l_{j+1}-1} [\delta_n(\theta(h(x))(n), \theta(h(e'_j(x)))(n)) + \delta_n(\theta(h(e_j(x)))(n), \theta(h(e'_j(x)))(n))]^q \\ \leq\ & 2^{q-1} \Big[\textstyle\sum_{n=l_j}^{l_{j+1}-1} \delta_n(\theta(h(x))(n), \theta(h(e'_j(x)))(n))^q \\ & \quad + \textstyle\sum_{n=l_j}^{l_{j+1}-1} \delta_n(\theta(h(e_j(x)))(n), \theta(h(e'_j(x)))(n))^q\Big] \\ \leq\ & 2^{q-1} \Big[\textstyle\sum_{n<l_{j+1}} \delta_n(\theta(h(x))(n), \theta(h(e'_j(x)))(n))^q \\ & \quad + \textstyle\sum_{n\geq l_j} \delta_n(\theta(h(e_j(x)))(n), \theta(h(e'_j(x)))(n))^q\Big] \\ <\ & 2^{q-1} \cdot 3 \cdot 2^{-j}. \end{aligned}$$

We can see that $\theta'(x) \restriction [l_j, l_{j+1}) = T_j(x(j)) = \theta(h(e_j(x))) \restriction [l_j, l_{j+1})$ for each $j \in \mathbb{N}$. Therefore,

$$\begin{aligned} & \textstyle\sum_{n\in\mathbb{N}} \delta_n(\theta(h(x))(n), \theta'(x)(n))^q \\ =\ & \textstyle\sum_{j\in\mathbb{N}} \sum_{n=l_j}^{l_{j+1}-1} \delta_n(\theta(h(x))(n), \theta'(x)(n))^q \\ =\ & \textstyle\sum_{j\in\mathbb{N}} \sum_{n=l_j}^{l_{j+1}-1} \delta_n(\theta(h(x))(n), \theta(h(e_j(x)))(n))^q \\ <\ & \textstyle\sum_{j\in\mathbb{N}} 2^{q-1} \cdot 3 \cdot 2^{-j} < +\infty, \end{aligned}$$

as desired. Claim (iii) $\square$

Note that

$$\begin{aligned}(\theta'(x),\theta'(\hat{x}))\in E((Y_n)_{n\in\mathbb{N}};q) \quad &\iff \textstyle\sum_{j\in\mathbb{N}}\sum_{n=l_j}^{l_{j+1}-1}\delta_n(\theta'(x)(n),\theta'(x)(n))^q<+\infty\\ &\iff \textstyle\sum_{j\in\mathbb{N}}\delta_q(T_j(x(j)),T_j(y(j)))^q<+\infty.\end{aligned}$$

This completes the proof. $\square$

Let (X,d) be a metric space and $C>0$. We consider the following condition:

(link(C)) *For $\varepsilon>0$, there exists $N\geq 1$ such that, for any $u,v\in X$ with $d(u,v)<C$, we can find $r_i\in X$, $i=0,1,\cdots,N$ with $r_0=u,r_N=v$ and $d(r_{i-1},r_i)<\varepsilon$ for each $i\geq 1$.*

Let (X,d) and (Y_n,δ_n), $n\in\mathbb{N}$ be separable complete metric spaces, $p,q\in[1,+\infty)$. Assume that

(A1) X satisfies (link(C)) for some $C>0$; and
(A2) $F(X;p)\leq_B E((Y_n)_{n\in\mathbb{N}};q)$.

Fix a sequence of finite subsets $F_n\subseteq X,n\in\mathbb{N}$ such that

$$F_0\subseteq F_1\subseteq\cdots\subseteq F_n\subseteq\cdots$$

and $\bigcup_{n\in\mathbb{N}}F_n$ is dense in X.

Since (link(C)) holds, for $l\in\mathbb{N}$, there exists $N(l)\geq 1$ such that, for any $u,v\in X$ with $d(u,v)<C$, we can find $r_i^l(u,v)\in X$, $i=0,1,\cdots,N(l)$ with $r_0^l(u,v)=u,r_{N(l)}^l(u,v)=v$ and $d(r_{i-1}^l(u,v),r_i^l(u,v))<2^{-l}$ for $i=1,\cdots,N(l)$. We denote

$$Z_n=\{r_i^l(u,v):u,v\in F_n,d(u,v)<C,l\leq n,i=0,1,\cdots,N(l)\}.$$

Note that $E((Z_n);p)\sim_B F(X;p)\leq_B E((Y_n)_{n\in\mathbb{N}};q)$. Since $Z_n\subseteq X$ is a sequence of finite metric spaces, we can find $(b_j)_{j\in\mathbb{N}},(l_j)_{j\in\mathbb{N}}$ and $T_j:Z_{b_j}\to\prod_{n=l_j}^{l_{j+1}-1}Y_n$ as in Lemma 4.1. Then we have the following lemmas.

Lemma 4.2. *For any $C'>0$, there exists a $D>0$ such that, for sufficiently large j and $u,v\in F_{b_j}$, if $d(u,v)\geq C'$, then $\delta_q(T_j(u),T_j(v))\geq D$.*

Proof. Assume for contradiction that, there exists a strictly increasing sequence of natural numbers $(j_k)_{k\in\mathbb{N}}$ such that there are $u_k,v_k\in F_{b_{j_k}}$ with $d(u_k,v_k)\geq C'$ and $\delta_q(T_{j_k}(u_k),T_{j_k}(v_k))<2^{-k}$.

Now we select $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$ such that

$$\begin{cases} x(j) = u_k, y(j) = v_k, & j = j_k, \\ x(j) = y(j), & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{j\in\mathbb{N}} d(x(j), y(j))^p = \sum_{k\in\mathbb{N}} d(u_k, v_k)^p \geq \sum_{k\in\mathbb{N}} (C')^p = +\infty,$$

so $(x, y) \notin E((Z_{b_j})_{j\in\mathbb{N}}; p)$. On the other hand, we have

$$\begin{aligned} \textstyle\sum_{j\in\mathbb{N}} \delta_q(T_j(x(j)), T_j(y(j)))^q &= \textstyle\sum_{k\in\mathbb{N}} \delta_q(T_{j_k}(u_k), T_{j_k}(v_k))^q \\ &< \textstyle\sum_{k\in\mathbb{N}} 2^{-kq} \\ &< +\infty, \end{aligned}$$

contradicting Lemma 4.1! □

Lemma 4.3. *There exists an $m \in \mathbb{N}$ such that $\forall k \exists N \forall j > N$, for $u, v \in F_{b_j}$, if $k^{-1} \leq d(u, v) < C$, then we have*

$$2^{-m} d(u, v)^{\frac{p}{q}} \leq \delta_q(T_j(u), T_j(v)) \leq 2^m d(u, v)^{\frac{p}{q}}.$$

Proof. Assume for contradiction that, for every m, $\exists k_m \exists^\infty j \exists u_j, v_j \in F_{b_j}$ such that $k_m^{-1} \leq d(u_j, v_j) < C$ but either

$$2^{-m} d(u_j, v_j)^{\frac{p}{q}} > \delta_q(T_j(u_j), T_j(v_j))$$

or

$$\delta_q(T_j(u_j), T_j(v_j)) > 2^m d(u_j, v_j)^{\frac{p}{q}}.$$

We define two subsets $I_1, I_2 \subseteq \mathbb{N}$. For $m \in \mathbb{N}$, we put $m \in I_1$ iff $\exists k_m \exists^\infty j \exists u_j, v_j \in F_{b_j}$ satisfying that $k_m^{-1} \leq d(u_j, v_j) < C$ and

$$2^{-m} d(u_j, v_j)^{\frac{p}{q}} > \delta_q(T_j(u_j), T_j(v_j));$$

and $m \in I_2$ iff $\exists k_m \exists^\infty j \exists u_j, v_j \in F_{b_j}$ satisfying that $k_m^{-1} \leq d(u_j, v_j) < C$ and

$$\delta_q(T_j(u_j), T_j(v_j)) > 2^m d(u_j, v_j)^{\frac{p}{q}}.$$

From the assumption, we can see that $I_1 \cup I_2 = \mathbb{N}$. Now we consider the following two cases.

Case 1. $|I_1| = \infty$. Select a finite set $J^m \subseteq \mathbb{N}$ for every $m \in I_1$ and $u_j, v_j \in F_{b_j}$ for $j \in J^m$ satisfying that

(i) for $j \in J^m$, we have $2^{-m} d(u_j, v_j)^{\frac{p}{q}} > \delta_q(T_j(u_j), T_j(v_j))$;
(ii) $C^p \le \sum_{j \in J^m} d(u_j, v_j)^p < 2C^p$;
(iii) if $m_1 < m_2$, then $\max J^{m_1} < \min J^{m_2}$.

Now we select $x, y \in \prod_{j \in \mathbb{N}} Z_{b_j}$ such that

$$\begin{cases} x(j) = u_j, y(j) = v_j, & j \in J^m, m \in I_1, \\ x(j) = y(j), & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{j \in \mathbb{N}} d(x(j), y(j))^p = \sum_{m \in I_1} \sum_{j \in J^m} d(u_j, v_j)^p \ge \sum_{m \in I_1} C^p = +\infty,$$

so $(x, y) \notin E((Z_{b_j})_{j \in \mathbb{N}}; p)$. On the other hand, we have

$$\begin{aligned} \sum_{j \in \mathbb{N}} \delta_q(T_j(x(j)), T_j(y(j)))^q &= \sum_{m \in I_1} \sum_{j \in J^m} \delta_q(T_j(u_j), T_j(v_j))^q \\ &< \sum_{m \in I_1} \sum_{j \in J^m} 2^{-mq} d(u_j, v_j)^p \\ &< 2C^p \sum_{m \in I_1} (2^{-q})^m \\ &< +\infty, \end{aligned}$$

contradicting Lemma 4.1!

Case 2. $|I_2| = \infty$. We can find a strictly increasing sequence of natural numbers $m_l \in I_2$, $l \in \mathbb{N}$ such that $m_l \ge \frac{pl}{2q}$ and $2^{m_l} \ge N(l)$ for each l.

We define two subsets $L_1, L_2 \subseteq \mathbb{N}$. For $l \in \mathbb{N}$, we put $l \in L_1$ iff $\exists^\infty j \exists u_j, v_j \in F_{b_j}$ satisfying that $k_{m_l}^{-1} \le d(u_j, v_j) < (\sqrt{2})^{-l}$ and

$$\delta_q(T_j(u_j), T_j(v_j)) > 2^{m_l} d(u_j, v_j)^{\frac{p}{q}};$$

and $l \in L_2$ iff $\exists^\infty j \exists u_j, v_j \in F_{b_j}$ satisfying that $(\sqrt{2})^{-l} \le d(u_j, v_j) < C$ and

$$\delta_q(T_j(u_j), T_j(v_j)) > 2^{m_l} d(u_j, v_j)^{\frac{p}{q}}.$$

Since each $m_l \in I_2$, we have $L_1 \cup L_2 = \mathbb{N}$. We consider two subcases.

Subcase 2.1. $|L_1| = \infty$. Select a finite set $K_1^l \subseteq \mathbb{N}$ for every $l \in L_1$ and $u_j, v_j \in F_{b_j}$ for $j \in K_1^l$ satisfying that

(i) for $j \in K_1^l$, we have $\delta_q(T_j(u_j), T_j(v_j)) > 2^{m_l} d(u_j, v_j)^{\frac{p}{q}}$;
(ii) $(\sqrt{2})^{-pl} \le \sum_{j \in K_1^l} d(u_j, v_j)^p < 2(\sqrt{2})^{-pl}$;
(iii) if $l_1 < l_2$, then $\max K_1^{l_1} < \min K_1^{l_2}$.

Now we select $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$ such that

$$\begin{cases} x(j) = u_j, y(j) = v_j, & j \in K_1^l, l \in L_1, \\ x(j) = y(j), & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{j\in\mathbb{N}} d(x(j), y(j))^p = \sum_{l\in L_1} \sum_{j\in K_1^l} d(u_j, v_j)^p \le 2 \sum_{l\in L_1} (\sqrt{2})^{-pl} < +\infty,$$

so $(x, y) \in E((Z_{b_j})_{j\in\mathbb{N}}; p)$. On the other hand, since $m_l \ge \frac{pl}{2q}$, we have

$$\begin{aligned} \textstyle\sum_{j\in\mathbb{N}} \delta_q(T_j(x(j)), T_j(y(j)))^q & = \textstyle\sum_{l\in L_1} \sum_{j\in K_1^l} \delta_q(T_j(u_j), T_j(v_j))^q \\ & > \textstyle\sum_{l\in L_1} \sum_{j\in K_1^l} 2^{qm_l} d(u_j, v_j)^p \\ & \ge \textstyle\sum_{l\in L_1} (\sqrt{2})^{2qm_l - pl} \\ & = +\infty, \end{aligned}$$

contradicting Lemma 4.1!

Subcase 2.2 $|L_2| = \infty$. Select a finite set $K_2^l \subseteq \mathbb{N}$ for each $l \in L_2$ and $u_j, v_j \in F_{b_j}$ for $j \in K_2^l$ satisfying that

- (i) for $j \in K_2^l$, we have $(\sqrt{2})^{-l} \le d(u_j, v_j) < C$ and $\delta_q(T_j(u_j), T_j(v_j)) > 2^{m_l} d(u_j, v_j)^{\frac{p}{q}}$;
- (ii) $C^p \le \sum_{j\in K_2^l} d(u_j, v_j)^p < 2C^p$;
- (iii) if $l_1 < l_2$, then $\max K_2^{l_1} < \min K_2^{l_2}$;
- (iv) for $j \in K_2^l$, we have $l \le b_j$.

For $l \in L_1$ and $j \in K_2^l$, since $d(u_j, v_j) < C$ and $l \le b_j$, by the definition of Z_{b_j} we have

$$r_i^l(u_j, v_j) \in Z_{b_j} \quad (i = 0, 1, \cdots, N(l)).$$

Since $r_0^l(u_j, v_j) = u_j, r_{N(l)}^l(u_j, v_j) = v_j$, the triangle inequality gives

$$\sum_{1\le i\le N(l)} \delta_q(T_j(r_{i-1}^l(u_j, v_j)), T_j(r_i^l(u_j, v_j))) \ge \delta_q(T_j(u_j), T_j(v_j)),$$

thus there is an $i(j)$ such that

$$\delta_q(T_j(r_{i(j)-1}^l(u_j, v_j)), T_j(r_{i(j)}^l(u_j, v_j))) \ge N(l)^{-1} \delta_q(T_j(u_j), T_j(v_j)).$$

Now denote $r_j = r^l_{i(j)-1}(u_j, v_j), s_j = r^l_{i(j)}(u_j, v_j)$. We select $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$ such that

$$\begin{cases} x(j) = r_j, y(j) = s_j, & j \in K^l_2, l \geq 1, \\ x(j) = y(j), & \text{otherwise.} \end{cases}$$

Note that $d(r_j, s_j) < 2^{-l} \leq (\sqrt{2})^{-l} d(u_j, v_j)$, we have

$$\begin{aligned} \sum_{j\in\mathbb{N}} d(x(j), y(j))^p &= \sum_{l\in L_2} \sum_{j\in K^l_2} d(r_j, s_j)^p \\ &< \sum_{l\in L_2} \sum_{j\in K^l_2} (\sqrt{2})^{-pl} d(u_j, v_j)^p \\ &< 2C^p \sum_{l\in L_2} (\sqrt{2})^{-pl} \\ &< +\infty, \end{aligned}$$

so $(x, y) \in E((Z_{b_j})_{j\in\mathbb{N}}, p)$. On the other hand, since $2^{m_l} \geq N(l)$ we have

$$\begin{aligned} \sum_{j\in\mathbb{N}} \delta_q(T_j(x(j)), T_j(y(j)))^q &= \sum_{l\in L_2} \sum_{j\in K^l_2} \delta_q(T_j(r_j), T_j(s_j))^q \\ &\geq \sum_{l\in L_2} \sum_{j\in K^l_2} N(l)^{-q} \delta_q(T_j(u_j), T_j(v_j))^q \\ &> \sum_{l\in L_2} \sum_{j\in K^l_2} N(l)^{-q} 2^{qm_l} d(u_j, v_j)^p \\ &\geq \sum_{l\in L_2} C^p \left(\frac{2^{m_l}}{N(l)}\right)^q \\ &= +\infty, \end{aligned}$$

contradicting Lemma 4.1 again! $\square$

Definition 4.4. *For two metric spaces $(X, d), (X', d')$ and $C, \alpha > 0$. We say that X can C-finitely Hölder(α) embed into X' if there exists $A, D > 0$ such that for every finite subset $F \subseteq X$, there is $T_F : F \to X'$ satisfying, for $u, v \in F$,*

(1) $d(u, v) \geq C \Rightarrow d'(T_F(u), T_F(v)) \geq D$;
(2) $d(u, v) < C \Rightarrow A^{-1} d(u, v)^\alpha \leq d'(T_F(u), T_F(v)) \leq A d(u, v)^\alpha$.

While $\alpha = 1$, we also say that X can C-finitely Lipschitz embed into X'.

Theorem 4.5. *Let (X, d) and (Y_n, δ_n), $n \in \mathbb{N}$ be separable complete metric spaces, $p, q \in [1, +\infty)$. If X satisfies $(\mathrm{link}(C))$ for some $C > 0$, and $F(X; p) \leq_B E((Y_n)_{n\in\mathbb{N}}; q)$, then X can C-finitely Hölder($\frac{p}{q}$) embed into $\ell_q((Y_n)_{n\in\mathbb{N}}, y^*)$ for any $y^* \in \prod_{n\in\mathbb{N}} Y_n$.*

Proof. Fix a sequence of finite subsets $F_n \subseteq X, n \in \mathbb{N}$ such that

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

and $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X. Let $(b_j)_{j\in\mathbb{N}}, (l_j)_{j\in\mathbb{N}}$ and $T_j : F_{b_j} \to \prod_{n=l_j}^{l_{j+1}-1} Y_n$ be from the remarks before Lemma 4.2. For convenience, we identify $(\prod_{n=l_j}^{l_{j+1}-1} Y_n, \delta_q)$ with a subspace of $\ell_q((Y_n)_{n\in\mathbb{N}}, y^*)$. Then T_j becomes a map $F_{b_j} \to \ell_q((Y_n)_{n\in\mathbb{N}}, y^*)$.

Let us consider an arbitrary finite subset $F \subseteq X$. We can find $k \in \mathbb{N}$ such that

(a) $k^{-1} \leq d(u, v)$ for any distinct $u, v \in F$;
(b) $d(u, v) \leq C - k^{-1}$ for any $u, v \in F$ with $d(u, v) < C$.

For every $u \in F$, since $\bigcup_{j\in\mathbb{N}} F_{b_j}$ is dense in X, there exists an $R(u) \in \bigcup_{j\in\mathbb{N}} F_{b_j}$ such that $d(u, R(u)) < (4k)^{-1}$. Then for any distinct $u, v \in F$, we have

$$d(R(u), R(v)) < d(u, v) + (2k)^{-1} \leq 2d(u, v),$$

and

$$d(R(u), R(v)) > d(u, v) - (2k)^{-1} \geq \frac{1}{2}d(u, v).$$

From Lemmas 4.2 and 4.3, there exist $D > 0, m \in \mathbb{N}$ and a sufficiently large i such that $R(u) \in F_{b_i}$ for every $u \in F$, and for $r, s \in F_{b_i}$,

(i) $d(r, s) \geq C - (2k)^{-1} \Rightarrow \delta_q(T_i(r), T_i(s)) \geq D$;
(ii) $(2k)^{-1} \leq d(r, s) < C \Rightarrow 2^{-m}d(r, s)^{\frac{p}{q}} \leq \delta_q(T_i(r), T_i(s)) \leq 2^m d(r, s)^{\frac{p}{q}}$.

We define $T_F : F \to \ell_q((Y_n)_{n\in\mathbb{N}}, y^*)$ by $T_F(u) = T_i(R(u))$ for $u \in F$.

For any $u, v \in F$ with $d(u, v) \geq C$, we have $d(R(u), R(v)) \geq C - (2k)^{-1}$. Then

$$\delta_q(T_F(u), T_F(v)) = \delta_q(T_i(R(u)), T_i(R(v))) \geq D.$$

For any distinct $u, v \in F$ with $d(u, v) < C$, we have $k^{-1} \leq d(u, v) \leq C - k^{-1}$. So $(2k)^{-1} \leq d(R(u), R(v)) \leq C - (2k)^{-1} < C$. Then

$$\begin{aligned} \delta_q(T_F(u), T_F(u)) &= \delta_q(T_i(R(u)), T_i(R(v))) \\ &\leq 2^m d(R(u), R(v))^{\frac{p}{q}} \\ &< 2^{m+\frac{p}{q}} d(u, v)^{\frac{p}{q}}, \end{aligned}$$

and

$$\begin{aligned} \delta_q(T_F(u), T_F(u)) &= \delta_q(T_i(R(u)), T_i(R(v))) \\ &\geq 2^{-m} d(R(u), R(v))^{\frac{p}{q}} \\ &> 2^{-(m+\frac{p}{q})} d(u, v)^{\frac{p}{q}}. \end{aligned}$$

Thus $A = 2^{m+\frac{p}{q}}$ and D witness that X can C-finitely Hölder($\frac{p}{q}$) embed into $\ell_q((Y_n)_{n\in\mathbb{N}}, y^*)$. □

Theorem 4.6. *Let $(X,d),(Y,\delta)$ be two separable complete metric spaces, $p,q\in[1,+\infty)$, and let $Y_0\subseteq Y_1\subseteq Y_2\subseteq\cdots$ be a sequence of Borel subsets of Y with $\bigcup_{n\in\mathbb{N}}Y_n$ dense in Y. If X satisfies* (link(C)) *for some $C>0$, then the following conditions are equivalent:*

(a) X can C-finitely Hölder($\frac{p}{q}$) embed into $\ell_q((Y_n)_{n\in\mathbb{N}},y^*)$ for some $y^*\in\prod_{n\in\mathbb{N}}Y_n$.
(b) $F(X;p)\leq_B E((Y_n)_{n\in\mathbb{N}};q)$.
(c) $F(X;p)\leq_B F(Y;q)$.

Proof. Let $F_0\subseteq F_1\subseteq F_2\subseteq\cdots$ be a sequence of finite subsets of X with $\bigcup_{n\in\mathbb{N}}F_n$ dense in X.

(a)⇒(b). Since X can C-finitely Hölder($\frac{p}{q}$) embed into $\ell_q((Y_n)_{n\in\mathbb{N}},y^*)$, we can find $A,D>0$, $T_n:F_n\to\ell_q((Y_n)_{n\in\mathbb{N}},y^*)$ such that, for $u,v\in F_n$,

(1) $d(u,v)\geq C\Rightarrow\delta_q(T_n(u),T_n(v))\geq D$;
(2) $d(u,v)<C\Rightarrow A^{-1}d(u,v)^{\frac{p}{q}}\leq\delta_q(T_n(u),T_n(v))\leq Ad(u,v)^{\frac{p}{q}}$.

Then $F(X;p)\sim_B E((F_n)_{n\in\mathbb{N}};p)\leq_B E((Y_n)_{n\in\mathbb{N}};q)$ follows from Theorem 2.3.

(b)⇒(a) follows from Theorem 4.5.

(b)⇒(c). Let $(b_j)_{j\in\mathbb{N}},(l_j)_{j\in\mathbb{N}}$ and $T_j:F_{b_j}\to\prod_{n=l_j}^{l_{j+1}-1}Y_n$ be from the remarks before Lemma 4.2. Since every F_{b_j} is finite, we can find finite subsets $U_n\subseteq Y_n$ for $l_j\leq n<l_{j+1}$ such that $T_j(u)\in\prod_{n=l_j}^{l_{j+1}-1}U_n$ for each $u\in F_{b_j}$. We can extend every U_n to a finite subset $W_n\subseteq Y$ such that $U_n\subseteq W_n$, $W_0\subseteq W_1\subseteq W_2\subseteq\cdots$ and $\bigcup_{n\in\mathbb{N}}W_n$ is dense in Y.

From Lemma 4.2 with $C'=C$ and Lemma 4.3 with $k=2^l$, we can find $D>0$, $m\in\mathbb{N}$ and a strictly increasing sequence of natural numbers $(j_l)_{l\in\mathbb{N}}$ such that, for $r,s\in F'_l\overset{\text{Def}}{=}F_{b_{j_l}}$, we have

(i) $d(r,s)\geq C\Rightarrow\delta_q(T_{j_l}(r),T_{j_l}(s))\geq D$;
(ii) $2^{-l}\leq d(r,s)<C\Rightarrow 2^{-m}d(r,s)^{\frac{p}{q}}\leq\delta_q(T_{j_l}(r),T_{j_l}(s))\leq 2^md(r,s)^{\frac{p}{q}}$.

Then Corollary 2.4 gives

$$F(X;p)\sim_B E((F'_l)_{l\in\mathbb{N}};p)\leq_B E((W_n)_{n\in\mathbb{N}};q)\sim_B F(Y;q).$$

(c)⇒(b). Find a sequence of finite subsets $V_n\subseteq Y_n$, $n\in\mathbb{N}$ such that $V_0\subseteq V_1\subseteq V_2\subseteq\cdots$ and $\bigcup_{n\in\mathbb{N}}V_n$ is dense in Y. Then we have $F(X;p)\leq_B F(Y;q)\sim_B E((V_n)_{n\in\mathbb{N}};q)\leq_B E((Y_n)_{n\in\mathbb{N}};q)$. □

Corollary 4.7. *Let X, Y be two separable complete metric spaces, $p, q \in [1, +\infty)$. If X satisfies* (link(C)) *for some $C > 0$, then the following conditions are equivalent:*

(a) X can C-finitely Hölder($\frac{p}{q}$) embed into $\ell_q(Y, y^*)$ for some $y^* \in Y^{\mathbb{N}}$.
(b) $F(X; p) \leq_B E(Y; q)$.
(c) $F(X; p) \leq_B F(Y; q)$.

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